

# SUPERCHARACTERS AND PATTERN SUBGROUPS IN THE UPPER TRIANGULAR GROUPS

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ABSTRACT. Let  $U_n(q)$  denote the upper triangular group of degree  $n$  over the finite field  $\mathbb{F}_q$  with  $q$  elements. It is known that irreducible constituents of supercharacters partition the set of all irreducible characters  $\text{Irr}(U_n(q))$ . In this paper we present a correspondence between supercharacters and pattern subgroups of the form  $U_k(q) \cap {}^w U_k(q)$  where  $w$  is a monomial matrix in  $GL_k(q)$  for some  $k < n$ .

## 1. INTRODUCTION

Let  $q$  be a power of a prime  $p$  and  $\mathbb{F}_q$  a field with  $q$  elements. The group  $U_n(q)$  of all upper triangular  $(n \times n)$ -matrices over  $\mathbb{F}_q$  with all diagonal entries equal to 1 is a Sylow  $p$ -subgroup of  $GL_n(\mathbb{F}_q)$ . It is conjectured by Higman [8] that the number of conjugacy classes of  $U_n(q)$  is given by a polynomial in  $q$  with integer coefficients. Isaacs [10] shows that the degrees of all irreducible characters of  $U_n(q)$  are powers of  $q$ . Huppert [9] proves that character degrees of  $U_n(q)$  are precisely of the form  $\{q^e : 0 \leq e \leq \mu(n)\}$  where the upper bound  $\mu(n)$  was known to Lehrer [14]. Lehrer [14] conjectures that each number  $N_{n,e}(q)$  of irreducible characters of  $U_n(q)$  of degree  $q^e$  is given by a polynomial in  $q$  with integer coefficients. Isaacs [11] suggests a strengthened form of Lehrer's conjecture stating that  $N_{n,e}(q)$  is given by a polynomial in  $(q-1)$  with nonnegative integer coefficients. So, Isaacs' conjecture implies Higman's and Lehrer's conjectures.

Many efforts have been made to understand more about  $U_n(q)$ , see Thompson [17], Robinson [16], André [1], Isaacs [10, 11], Diaconis-Isaacs [5], Arregi-Vera-López [3], Evseev [7]... Supercharacters arise as tensor products of some elementary characters to give a “nice” partition of all non-principal irreducible characters of  $U_n(q)$ , see [1] or [13]. Supercharacters have been defined for Sylow  $p$ -subgroups of other finite groups of Lie type, see [2]. Here, for  $U_n(q)$  we show a correspondence between supercharacters and pattern subgroups, Theorem 1.7. And we use it to decompose certain supercharacters into irreducible constituents in Application Section.

Let  $\Sigma = \Sigma_{n-1} = \langle \alpha_1, \dots, \alpha_{n-1} \rangle$  be the root system of  $GL_n(q)$  with respect to the maximal split torus equal to the diagonal group, see Chapter 3, [4]. Denote  $\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \dots + \alpha_j$  for all  $0 < i \leq j < n$ . Denote by  $\Sigma^+$  the set of all positive roots. The root subgroup  $X_{\alpha_{i,j}}$  is the set of all matrices of the form  $I_n + c \cdot e_{i,j+1}$ , where  $I_n$  = the identity  $(n \times n)$ -matrix,  $c \in \mathbb{F}_q$  and  $e_{i,j+1}$  = the zero matrix except 1 at entry  $(i, j+1)$ . The upper triangular group  $U_n(q)$  is generated by all  $X_\alpha$  where  $\alpha \in \Sigma^+$ . We write  $U$  for  $U_n(q)$  if  $n$  and  $q$  are clear from the context. To be

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*Date:* March 2nd, 2010.

*2000 Mathematics Subject Classification.* Primary 20C33, 20C15.

*Key words and phrases.* Tensor product, root system, irreducible characters, representations.

convenient for using the root system, we consider the upper triangular group as a tableaux.

$$\begin{pmatrix} 1 & * & * & * & * \\ . & 1 & * & * & * \\ . & . & 1 & * & * \\ . & . & . & 1 & * \\ . & . & . & . & 1 \end{pmatrix} \rightarrow \begin{array}{|c|c|c|c|} \hline \alpha_1 & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \hline & \alpha_2 & \alpha_{2,3} & \alpha_{2,4} \\ \hline & & \alpha_3 & \alpha_{3,4} \\ \hline & & & \alpha_4 \\ \hline \end{array}$$

A subset  $S \subset \Sigma^+$  is called *closed* if for each  $\alpha, \beta \in S$  such that  $\alpha + \beta \in \Sigma^+$  then  $\alpha + \beta \in S$ . A *pattern* subgroup of  $U$  is a group generated by all root subgroups  $X_\alpha$ , where  $\alpha \in S$  a closed positive root subset.

Let  $G$  be a group. Denote  $G^\times = G \setminus \{1\}$ ,  $\text{Irr}(G)$  the set of all complex irreducible characters of  $G$ , and  $\text{Irr}(G)^\times = \text{Irr}(G) \setminus \{1_G\}$ . For  $H \trianglelefteq G$ , let  $\text{Irr}(G/H)$  denote the set of all irreducible characters of  $G$  with  $H$  in the kernel. If  $K \leq G$  such that  $G = H \rtimes K$ , then for each character  $\xi$  of  $K$ , we denote the inflation of  $\xi$  to  $G$  by  $\xi_G$ , i.e.  $\xi_G$  is the extension of  $\xi$  to  $G$  with  $H \subset \ker(\xi_G)$ . Furthermore, for  $H \leq G$  and  $\xi \in \text{Irr}(H)$ , we define  $\text{Irr}(G, \xi) = \{\chi \in \text{Irr}(G) : (\chi, \xi^G) \neq 0\}$  the irreducible constituent set of  $\xi^G$ , and for  $\chi \in \text{Irr}(G)$ , we denote its restriction to  $H$  by  $\chi|_H$ .

For a field  $K$ , let  $K^\times$  be its multiplicative group. In the whole paper, we fix a nontrivial linear character  $\varphi : (\mathbb{F}_q, +) \rightarrow \mathbb{C}^\times$ . For each  $\alpha \in \Sigma^+$  and  $s \in \mathbb{F}_q$ , the map  $\phi_{\alpha,s} : X_\alpha \rightarrow \mathbb{C}^\times, x_\alpha(d) \mapsto \varphi(ds)$  is a linear character of the root subgroup  $X_\alpha$ , and all linear characters of  $X_\alpha$  arise in this way.

For each  $\alpha_{i,j}$ , define  $\text{arm}(\alpha_{i,j}) = \{\alpha_{i,k} : i \leq k < j\}$  and  $\text{leg}(\alpha_{i,j}) = \{\alpha_{k,j} : i < k \leq j\}$ . If  $i = j$ ,  $\alpha_{i,i} = \alpha_i$ , then  $\text{arm}(\alpha_i)$  and  $\text{leg}(\alpha_i)$  are empty. For each  $\alpha \in \Sigma^+$ , define the *hook* of  $\alpha$  as  $h(\alpha) = \text{arm}(\alpha) \cup \text{leg}(\alpha) \cup \{\alpha\}$ , the *hook group* of  $\alpha$  as  $H_\alpha = \langle X_\beta : \beta \in h(\alpha) \rangle$ , and the *base group*  $V_\alpha = \langle X_\beta : \beta \in \Sigma^+ \setminus \text{arm}(\alpha) \rangle$ . Since  $[V_\alpha, V_\alpha] \cap X_\alpha = \{1\}$ , for each  $s \in \mathbb{F}_q^\times$  there exists a linear  $\lambda_{\alpha,s} \in \text{Irr}(V_\alpha)$  such that  $\lambda_{\alpha,s}|_{X_\alpha} = \phi_{\alpha,s}$  and  $\lambda_{\alpha,s}|_{X_\beta} = 1_{X_\beta}$  for the others  $X_\beta \subset V_\alpha$ ,  $\beta \neq \alpha$ . Denote by  $\text{Irr}(V_\alpha/[V_\alpha, V_\alpha])^\times$  the set of all these linear characters of  $V_\alpha$ .

**Lemma 1.1.**  $\lambda_{\alpha,s}^U$  is irreducible for all  $s \in \mathbb{F}_q^\times$ .

*Proof.* See Lemma 2, [1] or Lemma 2.2, [13].  $\square$

We call  $\lambda_{\alpha,s}^U$  an *elementary* character of  $U$  associated to  $\alpha$ . A *basic* set  $D$  is a nonempty subset of  $\Sigma^+$  in which none of roots are on the same row or column. For each basic set  $D$ , denote  $E(D) = \bigoplus_{\alpha \in D} \text{Irr}(V_\alpha/[V_\alpha, V_\alpha])^\times$ . For each basic set  $D$  and  $\phi \in E(D)$ , we define a *supercharacter*, also known as *basic* character in [1],

$$\xi_{D,\phi} = \bigotimes_{\lambda_{\alpha,s} \in \phi} \lambda_{\alpha,s}^U.$$

It turns out that each supercharacter  $\xi_{D,\phi}$  is induced from a linear character of a pattern subgroup.

**Definition 1.2.** Define  $V_D = \bigcap_{\alpha \in D} V_\alpha$  and  $\lambda_D = \bigotimes_{\lambda_{\alpha,s} \in \phi} \lambda_{\alpha,s}|_{V_D}$ .

**Lemma 1.3.** We have  $\xi_{D,\phi} = \lambda_D^U$ .

*Proof.* See Lemma 2.5, [13].  $\square$

It is easy to see that  $V_D$  is generated by all  $X_\beta$  where  $\beta \in \Sigma^+ \setminus (\bigcup_{\alpha \in D} \text{arm}(\alpha))$ , and  $\lambda_D$  is a linear character of  $V_D$ . For each basic set  $D$ , it can be proved that the

diagonal subgroup of  $GL_n(q)$  acts transitively on  $E(D)$  by conjugation. So it makes sense when we write  $\lambda_D$  here instead of  $\lambda_{D,\phi}$ , and it also says that the decomposition of  $\xi_{D,\phi}$  is only dependent on  $D$ . To know more about supercharacters, see [5], [6]... Here, we recall the main role of supercharacters as a partition of  $\text{Irr}(U)^\times$ .

**Theorem 1.4.** *For each  $\chi \in \text{Irr}(U)^\times$ , there exist uniquely a basic set  $D$  and  $\phi \in E(D)$  such that  $\chi$  is an irreducible constituent of  $\xi_{D,\phi}$ .*

*Proof.* See Theorem 1, [1] or Theorem 2.6, [13].  $\square$

Denote by  $\text{Irr}(\xi_{D,\phi})$  the set of all irreducible constituents of  $\xi_{D,\phi}$ . Here, to prove Higman's conjecture, it suffices to prove that  $|\text{Irr}(\xi_{D,\phi})|$  is a polynomial in  $q$ .

Now for each basic set  $D$  of size  $k = |D|$ , we define an associated monomial  $(k \times k)$ -matrix  $w_D \in GL_k(q)$ . First of all, we define two partial orders on  $\Sigma^+$ .

**Definition 1.5.** *We define  $<_r$  and  $<_b$  on  $\Sigma^+$  as follows*

- (i)  $\alpha_{i,j} <_r \alpha_{l,k}$  if  $j < k$  (i.e. to the right)
- (ii)  $\alpha_{i,j} <_b \alpha_{l,k}$  if  $i < l$  (i.e. to the bottom).

An easy way to understand these two orders is  $<_r$  standing for left to right and  $<_b$  for top to bottom. It is noted that on a basic set,  $<_r$  and  $<_b$  are total orders.

Now we fix a basic set  $D$  of size  $k$  ascending order of  $<_r$ . Let  $D = \{\tau_1, \dots, \tau_k\}$  where  $\tau_i <_r \tau_j$  if  $i < j$ . We define  $w_D = (a_{i,j}) \in GL_k(q)$  as follows

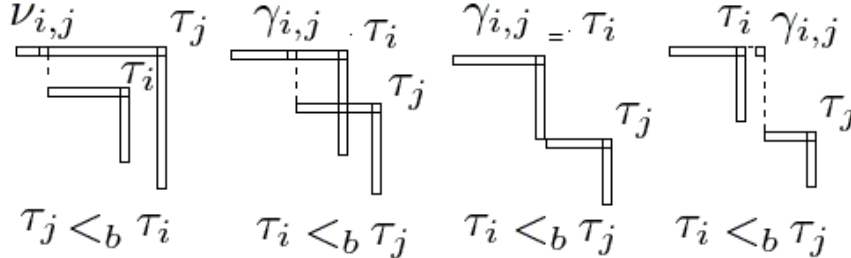
$$a_{i,j} = \begin{cases} 1 & \text{if } \tau_j \text{ is the } i\text{-th element of } D \text{ in ascending order } <_b, \\ 0 & \text{otherwise.} \end{cases}$$

For example, if  $D = \{\alpha_{2,3}, \alpha_{1,4}, \alpha_{3,5}\}$ ,  $|D| = 3$ ,

$$\begin{array}{ccccc} & & & \alpha_{1,4} & \\ & & & \alpha_{2,3} & \\ & & & & \alpha_{3,5} \\ & & & & \\ & & & & \end{array} \text{ then } w_D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is clear that  $w_D$  is a monomial matrix in the Weyl group  $S_k$  of  $GL_k(q)$ . Here,  $w_D$  somehow gives pivots of  $D$  by considering only rows and columns containing roots in  $D$ . Hence, it is equivalent to apply the (total) orders  $<_r$ ,  $<_b$  to these monomial matrices on their nonzero entries.

For each pair  $0 < i < j \leq k$ , if  $\tau_i <_b \tau_j$ , let  $\gamma_{i,j}$  be the root on the row of  $\tau_i$  such that  $\gamma_{i,j} + \tau_j \in \Sigma^+$ ; otherwise, i.e.  $\tau_j <_b \tau_i$ , let  $\nu_{i,j}$  be the root on the row of  $\tau_j$  such that  $\nu_{i,j} + \tau_i \in \Sigma^+$ . For example,  $\tau_i = \alpha_{m,i}$ ,  $\tau_j = \alpha_{l,j}$  where  $i < j$ , so if  $\alpha_{m,i} <_b \alpha_{l,j}$ , i.e.  $m < l$ , then  $\gamma_{i,j} = \alpha_{m,l-1}$ ; otherwise, if  $\alpha_{l,j} <_b \alpha_{m,i}$ , i.e.  $l < m$ , then  $\nu_{i,j} = \alpha_{l,m-1}$ . It is easy to see that  $\nu_{i,j}$  exists if and only if two hooks  $h(\tau_i)$  and  $h(\tau_j)$  are parallel, otherwise  $\gamma_{i,j}$  exists.



Let  $\Gamma_D$  be the set of all  $\gamma_{i,j}$ ,  $\Lambda_D$  the set of all  $\nu_{i,j}$ , and  $\Delta_D = \Gamma_D \cup \Lambda_D$ . Hence, by definitions for the existence of  $\gamma_{i,j}$  and  $\nu_{i,j}$ ,  $\Gamma_D \cap \Lambda_D = \emptyset$ . The next theorem provides a correspondence between supercharacters and pattern subgroups.

**Definition 1.6.** Define  $R_D = \langle X_\gamma \mid \gamma \in \Gamma_D \rangle$ ,  $C_D = \langle X_\nu \mid \nu \in \Lambda_D \rangle$ .

**Theorem 1.7.** Let  $\xi_{D,\phi}$  be a supercharacter. The following are true.

- (i)  $\xi_{D,\phi} = (\lambda_D^{\langle V_D, R_D \rangle})^U$ .
- (ii) For each  $\chi \in \text{Irr}(\langle V_D, R_D \rangle, \lambda_D)$ ,  $\chi^U \in \text{Irr}(\xi_{D,\phi})$ .
- (iii) If  $\chi_1 \neq \chi_2 \in \text{Irr}(\langle V_D, R_D \rangle, \lambda_D)$  then  $\chi_1^U \neq \chi_2^U$ .

Therefore, to decompose  $\xi_{D,\phi}$ , it suffices to decompose  $\lambda_D^{\langle V_D, R_D \rangle}$ . The next lemma provides interesting correspondences between the size of  $D$  and  $\Delta_D$ , between  $w_D$  and  $\Gamma_D, \Lambda_D$ . Moreover, it shows that  $\langle V_D, R_D \rangle = V_D R_D$ , the induced character  $\lambda_D^{V_D R_D}$  is equivalent to a constituent of the regular character  $1^{R_D}$ , and the pattern subgroup  $R_D$  is only determined by  $w_D$  in a quite natural way.

**Lemma 1.8.** Let  $D$  be a basic set of size  $k$ . The following are true.

- (i)  $\Delta_D$  is closed and  $\langle R_D, C_D \rangle = \langle X_\alpha \mid \alpha \in \Delta_D \rangle$  is isomorphic to  $U_k(q)$ .
- (ii)  $\Gamma_D$  is closed. For each pair  $i < j$ , if  $\gamma_{i,s}, \gamma_{j,r}$  exist and  $\gamma_{i,s} + \gamma_{j,r} \in \Sigma^+$ , then  $s = j$  and  $\gamma_{i,j} + \gamma_{j,r} = \gamma_{i,r}$ .
- (iii)  $\Lambda_D$  is closed. For each pair  $i < j$ , if  $\nu_{i,s}, \nu_{j,r}$  exist and  $\nu_{i,s} + \nu_{j,r} \in \Sigma^+$ , then  $s = j$  and  $\nu_{i,j} + \nu_{j,r} = \nu_{i,r}$ .
- (iv)  $R_D$  is isomorphic to  $U_k(q) \cap {}^{w_D}U_k(q)$  and  $C_D$  is isomorphic to  $U_k(q) \cap {}^{w_0 w_D}U_k(q)$  where  $w_0 = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}$  the longest element in  $S_k$ .

- (v)  $V_D R_D$  is a pattern subgroup of  $U$  and  $R_D$  normalizes  $V_D$ .

For example, let  $D = \{\alpha_{1,2}, \alpha_{3,4}, \alpha_{4,5}, \alpha_{2,6}\}$  be a basic set in  $\Sigma_6^+$ .

$$U_7(q) = \begin{array}{|c|c|c|c|c|c|} \hline & \alpha_{1,2} & & & & \\ \hline & & & & & \alpha_{2,6} \\ \hline & & & & & \\ \hline & & & \alpha_{3,4} & & \\ \hline & & & & \alpha_{4,5} & \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array}$$

and

$$R_D = \begin{array}{|c|c|c|} \hline & \alpha_{1,2} & \\ \hline \end{array}, \quad C_D = \begin{array}{|c|c|} \hline & \\ \hline \end{array}$$

From Theorem 1.7 and Lemma 1.8 (v), we obtain a nice decomposition of  $\xi_{D,\phi}$ .

**Corollary 1.9.** Let  $\xi_{D,\phi}$  be a supercharacter. The following are true.

- (i)  $\text{Irr}(\xi_{D,\phi}) = \{\chi^U : \chi \in \text{Irr}(V_D R_D, \lambda_D)\}$ ,
- (ii)  $\xi_{D,\phi} = \sum_{\chi \in \text{Irr}(V_D R_D, \lambda_D)} \chi(1) \chi^U$ .

Theorem 1.4, Lemma 1.8 and Corollary 1.9 give a clear proof for the following corollary which is a different version of Theorem 1.4, [1].

**Corollary 1.10.**  $(\xi_{D,\phi}, \xi_{D',\phi'}) = \begin{cases} [V_D R_D : V_D] & \text{if } (D, \phi) = (D', \phi') \\ 0 & \text{otherwise.} \end{cases}$

As an application, we take  $U_{13}(q)$  as a sample. It was conjectured by Isaacs-Karagueuzian [12] that  $U_{13}(2)$  has a unique pair of irrational irreducible characters of degree  $2^{16}$ . This conjecture is solved with an affirmative answer by Marberg [15] and generalized by Evseev [7]. Here, using the approach of root system, we work independently to obtain representations and constructions of all irreducible constituents of the supercharacter of  $U_{13}(q)$ , which gives the irrational pair when  $q = 2$ . With the definition that a character is *well-induced* if it is induced from a linear character of some pattern subgroup by Evseev [7], this pair of irrational characters are not well-induced. Hence, it provides a more explainable script to the generalization of not well-induced characters. Finally, we list two families of supercharacters  $\xi_{D,\phi}$  which have exactly one irreducible constituent, i.e.  $\xi_{D,\phi} = m \cdot \chi$  for some  $\chi \in \text{Irr}(U)$ .

## 2. SUPERCHARACTERS AND PATTERN SUBGROUPS

In this section, we mainly prove Theorem 1.7 to give a correspondence between supercharacters  $\xi_{D,\phi}$  and pattern subgroups  $U_k(q) \cap {}^{w_D}U_k(q)$  where  $k = |D|$ . First of all, we are going to prove Lemma 1.8.

*Proof of Lemma 1.8.* Suppose that  $D = \{\tau_1, \dots, \tau_k\}$  in ascending order  $<_r$ .

(i) If we rearrange  $D$  in ascending order of  $<_b$  to be  $\{\theta_1, \dots, \theta_k\}$ , it is clear that on the row of  $\theta_i$ ,  $\Delta_D$  has  $(k-i)$  roots and the row of  $\theta_k$  does not have any root in  $\Delta_D$ .

For each pair  $i < j \in [1, k]$ , let  $\omega_{i,j} \in \Delta_D$  be the root on the row of  $\tau_i$  such that  $\omega_{i,j} + \tau_j \in \Sigma^+$ . (It is noted that  $\omega_{i,j}$  is either  $\gamma \in \Gamma_D$  or  $\nu \in \Lambda_D$ .) Hence, if  $\tau_i = \alpha_{i_1, i_2} <_b \tau_j = \alpha_{j_1, j_2}$ , i.e.  $i_1 < j_1$ , we have  $\omega_{i,j} = \alpha_{i_1, j_1-1}$ . Therefore, for each  $\omega_{i,j} = \alpha_{i_1, j_1-1} <_r \omega_{m,l} = \alpha_{m_1, l_1-1} \in \Delta_D$ , if  $\omega_{i,j} + \omega_{m,l} \in \Sigma^+$ , then it must be  $j_1 = m_1$ , and  $\omega_{i,j} + \omega_{j,l} = \alpha_{i_1, l_1-1} = \omega_{i,l}$ . This shows that  $\Delta_D$  is closed, and the longest root in  $\Delta_D$  is  $\omega_{1,2} + \dots + \omega_{k-1,k} = \omega_{1,k}$ . So  $\omega_{i,j}$  corresponds to  $\alpha_{i,j-1}$  in the positive root set  $\Sigma_{k-1}^+$ . Therefore,  $\langle X_\alpha \mid \alpha \in \Delta_D \rangle$  is a pattern subgroup isomorphic to  $U_k(q)$ .

(ii) With the same argument in (i), by the definition of  $\gamma_{i,s}$ ,  $\gamma_{j,r}$ , if  $\gamma_{i,s} + \gamma_{j,r} \in \Sigma^+$ , then  $s = j$ . By the transitive property of  $<_r, <_b$  on  $\tau_i, \tau_j, \tau_r$ , from  $\tau_i <_r, <_b \tau_j$  and  $\tau_j <_r, <_b \tau_r$ , we have  $\tau_i <_r, <_b \tau_r$ . So  $\gamma_{i,r}$  exists and  $\gamma_{i,j} + \gamma_{j,r} = \gamma_{i,r}$  follows.

(iii) The same argument of (ii) for  $\nu_{i,s}$  and  $\nu_{j,r} \in \Lambda_D$ .

(iv) Let  $w_D = (w_{i,j}) \in S_k \subset GL_k(q)$ . Since  $w_D$  is a monomial matrix,  $w_D^{-1} = w_D^t$ , the transpose of  $w_D$ . For each  $X = (x_{i,j}) \in U_k(q)$ , we observe  $Y := w_D \cdot X \cdot w_D^{-1}$ . Let  $Y = (y_{i,j})$ . For each pair  $i < j$ , we have  $y_{i,j} = \sum_{s,r \in [1,k]} w_{i,s} x_{s,r} w_{j,r}$ . Since  $i, j$  are fixed, there exist unique  $1 \leq f, h \leq k$  such that  $w_{i,f} = 1 = w_{j,h}$ , others  $w_{i,s} = 0 = w_{j,r}$ . Hence,  $y_{i,j} = w_{i,f} x_{f,h} w_{j,h}$ .

Since  $h \neq f$  and all  $x_{s,r} = 0$  if  $r < s$ , we have

$$\begin{aligned} & \text{if } f > h, \text{ i.e. } w_{i,f} <_b w_{j,h} \text{ and } w_{j,h} <_r w_{i,f}, y_{i,j} = 0, \text{ and} \\ & \text{if } f < h, \text{ i.e. } w_{i,f} <_b w_{j,h} \text{ and } w_{i,f} <_r w_{j,h}, y_{i,j} \text{ has nonzero values.} \end{aligned}$$

So  $R_D$  is isomorphic to  $U_k(q) \cap {}^{w_D}U_k(q)$  by the definition of  $\gamma_{i,j} \in \Gamma_D$ . And, hence,  $C_D$  is isomorphic to  $U_k(q) \cap {}^{w_0 \cdot w_D}U_k(q)$  by (i), (ii), (iii) and  $\Delta_D = \Gamma_D \cup \Lambda_D$ .

(v) From the definition of  $\gamma_{i,j}$ , it is easy to check that  $R_D$  normalizes  $V_D$ . Hence,  $V_D \cdot R_D$  is a pattern subgroup of  $U$ .  $\square$

Set  $K_D = \langle X_\alpha : X_\alpha \subset V_D \text{ and } \alpha \notin D \rangle = \langle X_\alpha : X_\alpha \subset V_D \cap \ker(\lambda_D) \rangle$ . It is clear that  $K_D$  is normal in  $V_D$ ,  $[V_D : K_D] = q^{|D|}$ , and  $V_D = K_D \cdot \prod_{\tau \in D} X_\tau$ . To prove Theorem 1.7, we need the following lemma.

**Lemma 2.1.** *Let  $\xi_{D,\phi}$  be a supercharacter. The following are true.*

- (i)  $K_D \subset \ker(\lambda_D^{V_D R_D})$ . Moreover,  $\lambda_D^{V_D R_D}(x) = [V_D R_D : V_D] \lambda_D(x)$  for all  $x \in V_D$ .
- (ii)  $(K_D \cap R_D) \trianglelefteq R_D$  and  $(V_D \cap R_D)/(K_D \cap R_D) \subset Z(R_D/(K_D \cap R_D))$ .
- (iii) Let  $\overline{\phi_D} = \{\lambda_{\alpha,s} \in \phi : X_\alpha \not\subseteq R_D\}$ . We have

$$\lambda_D^{V_D R_D} = (\lambda_D|_{V_D \cap R_D})^{R_D}_{V_D R_D} \otimes (\bigotimes_{\lambda_{\alpha,s} \in \overline{\phi_D}} (\lambda_{\alpha,s}|_{V_D})_{V_D R_D}).$$

*Proof.* (i) It is enough to show the statement for all  $X_\alpha \subset V_D$ . By Lemma 1.8 (v)  $V_D \trianglelefteq V_D R_D$ , we have  $\lambda_D^{V_D R_D}(x) = \frac{1}{|V_D|} \sum_{y \in V_D R_D} \lambda_D(x^y)$  for all  $x \in V_D$ . For each  $x \in X_\alpha$ , suppose that there is  $X_\beta \subset V_D R_D$  such that  $\alpha + \beta \in \Sigma^+$ , hence  $X_{\alpha+\beta} \subset V_D$ . We are going to show that  $\lambda_D(x^y) = \lambda_D(x)$  for all  $y \in X_\beta$ .

Since  $X_\tau \cap [V_D, V_D] = \{1\}$  for all  $\tau \in D$ , we have  $X_{\alpha+\beta} \subset K_D \subset \ker(\lambda_D)$ . Thus,  $[\lambda_D(x), \lambda_D(y)] = \lambda_D([x, y]) = 1$  since  $[x, y] \in X_{\alpha+\beta}$ , i.e.  $\lambda_D(x)^{-1} \lambda_D(x^y) = 1$ .

(ii) By the definition of  $K_D \trianglelefteq V_D$  and  $V_D = K_D \cdot \prod_{\tau \in D} X_\tau$ , it suffices to show that  $(K_D \cap R_D) \trianglelefteq R_D$ . This is clear because for all  $X_\alpha \subset K_D \cap R_D$  and all  $X_\beta \subset R_D$ , either  $\alpha + \beta \notin \Sigma^+$  or  $X_{\alpha+\beta} \subset K_D \cap R_D$ .

(iii) The inflations to  $V_D R_D$  of  $\lambda_D|_{V_D \cap R_D}^{R_D}$  and  $\lambda_{\alpha,s}|_{V_D}$ , for all  $\lambda_{\alpha,s} \in \overline{\phi_D}$ , come directly from (i).  $\square$

By Lemma 2.1 (iii), if  $R_D \cap V_D = \{1\}$ ,  $\lambda_D^{V_D R_D}$  is equivalent to  $1^{R_D}$ , the regular character of  $R_D$ . In general,  $\lambda_D^{V_D R_D}$  is equivalent to a constituent of  $1^{R_D}$  with  $R_D \cap K_D$  in the kernel. Now we prove Theorem 1.7.

*Proof of Theorem 1.7.* (i) It is clear by the transitivity of induction.

(ii) Suppose  $D = \{\tau_1, \dots, \tau_k\}$  in ascending order  $<_r$  and  $\lambda_D = \bigotimes_{\tau_i \in D} \lambda_{\tau_i, s_i}|_{V_D}$  where  $s_i \in \mathbb{F}_q^\times$ .

First, we show that each  $\chi \in \text{Irr}(V_D R_D, \lambda_D)$ ,  $\chi^U$  is irreducible. By the transitive property of induction, we are going to induce  $\chi$  from  $V_D R_D$  to  $U$  by a sequence of inductions along the arms of  $\tau_1, \tau_2, \dots, \tau_k$  respectively by  $<_r$  order. Now we setup these such induction steps.

For each  $\tau_i \in D$ , let  $A(\tau_i) = \{\alpha \in \text{arm}(\tau_i) : X_\alpha \not\subseteq V_D R_D\}$ , and  $c_i = |A(\tau_i)|$ . Let  $d_0 = 0$ , and  $d_i = d_{i-1} + c_i$  for all  $i \in [1, k]$ . Now if  $c_i > 0$ ,  $i \in [1, k]$ , we arrange  $A(\tau_i)$  in decreasing order  $<_r$  to be  $\{\beta_{d_{i-1}+1}, \dots, \beta_{d_{i-1}+c_i}\}$ . Let  $M_0 = V_D R_D$ ,  $M_{i+1} = M_i \rtimes X_{\beta_i}$  for all  $i \in [1, d_k]$ . It is clear that  $M_{d_k+1} = U$  and  $X_{\beta_j}$  normalizes  $M_j$ , hence this sequence of pattern subgroups is well-defined.

For each  $\beta_j \in \text{arm}(\tau_i)$ ,  $j \in [1, d_k]$ , there exists unique  $\delta \in \text{leg}(\tau_i)$  such that  $\beta_j + \delta = \tau_i$  and  $X_\delta \subset K_D$ , since if  $X_\delta \not\subseteq K_D$ , there exists  $\tau_m \in D$  such that  $\delta \in \text{arm}(\tau_m)$ , so  $\tau_i <_r \tau_m$ ,  $\tau_i <_b \tau_m$ , and this implies  $\beta_j = \gamma_{i,m}$ . We number this  $\delta$  as  $\delta_j$ , and let  $L(D) = \{\delta_j : j \in [1, d_k]\}$ . By Lemma 2.1 (i),  $X_\delta \subset \ker(\chi)$  for all  $\delta \in L(D)$ . Now we proceed the induction of  $\chi$  from  $V_D R_D$  to  $U$  via a sequence of pattern subgroups along the arms of all  $\tau_i \in D$ , namely from  $M_0$  to  $M_1, \dots, M_{d_k+1} = U$ .

Suppose that  $\chi^{M_j} \in \text{Irr}(M_j)$  for some  $M_j$ ,  $j \in [1, d_k+1]$  and  $X_{\delta_t} \subset \ker(\chi^L)$  for all  $t \in [j, d_k]$ . If  $j = d_k + 1$ , it is done. Otherwise, the next induction step is from

$M_j$  to  $M_{j+1} = M_j X_{\beta_j}$ , and suppose that it happens on the arm of  $\tau_i$ . For each  $x \in X_{\beta_j}^\times$ , since  $[X_{\delta_j}, x] = X_{\tau_i}$ , there is some  $y \in X_{\delta_j}$  such that  $\lambda_{\tau_i, s_i}([y, x]) \neq 1$  and

$${}^x(\chi^{M_j})(y) = \chi^{M_j}(y^x) = \chi^{M_j}([y, x]y) = \lambda_{\tau_i, s_i}([y, x])\chi^{M_j}(y) \neq \chi^{M_j}(y) = \chi^{M_j}(1).$$

Hence  $X_{\delta_j} \not\subseteq \ker({}^x(\chi^{M_j}))$ , and  ${}^x(\chi^{M_j}) \neq \chi^{M_j}$  for all  $x \in X_{\beta_j}^\times$ . This shows that the inertia group  $I_{M_j X_{\beta_j}}(\chi) = M_j$  and  $\chi^{M_j X_{\beta_j}} \in \text{Irr}(M_j X_{\beta_j}, \lambda_D)$ .

It is easy to check directly that  $X_{\delta_t} \subset \ker(\chi^{M_j X_{\beta_j}})$  for all  $t \in [j+1, d_k]$  by using  $[X_{\beta_j}, X_{\delta_t}] \subset \ker(\chi^{M_j})$ . Therefore, we have  $\chi^U$  is irreducible for all  $\chi \in \text{Irr}(V_D R_D, \lambda_D)$  by induction on  $j$ .

(iii) Now suppose  $\chi_1 \neq \chi_2 \in \text{Irr}(V_D R_D, \lambda_D)$  and  $\chi_1^{M_j} \neq \chi_2^{M_j}$  for some  $M_j$  as above, it is enough to show that  $\chi_1^{M_j X_{\beta_j}} \neq \chi_2^{M_j X_{\beta_j}}$ , where  $\beta_j \in \text{arm}(\tau_i)$ . It is noted that  $X_{\delta_j} \subset \ker(\chi_1^{M_j}) \cap \ker(\chi_2^{M_j})$ .

By Mackey formula with the double coset  $M_j \backslash M_j X_{\beta_j} / M_j$  represented by  $X_{\beta_j}$ ,  $(\chi_1^{M_j X_{\beta_j}}, \chi_2^{M_j X_{\beta_j}}) = \sum_{x \in X_{\beta_j}} (\chi_1^{M_j}, {}^x(\chi_2^{M_j}))$ . By using the same argument in (ii),  $X_{\delta_j} \not\subseteq \ker({}^x(\chi_2^{M_j}))$  for all  $x \in X_{\beta_j}^\times$ . Hence,  ${}^x(\chi_2^{M_j}) \neq \chi_1^{M_j}$  for all  $x \in X_{\beta_j}^\times$  since  $X_{\delta_j} \subset \ker(\chi_1^{M_j})$ . Therefore,  $(\chi_1^{M_j X_{\beta_j}}, \chi_2^{M_j X_{\beta_j}}) = (\chi_1^{M_j}, \chi_2^{M_j}) = 0$  since  $\chi_1^{M_j} \neq \chi_2^{M_j}$  by the above assumption on  $M_j$ .  $\square$

It is noted that  $V_D R_D$  is not normal in  $U$ . In the proof of Theorem 1.7, although all inductions from  $V_D R_D$  to  $U$  are irreducible, Clifford correspondence can not be applied. The technique of a sequence of inductions from  $M_j$  to  $M_{j+1} \subset N_U(M_j)$  has been used to control distinct induced characters.

Since  $V_D$  is normal in  $V_D R_D$  and  $V_D R_D / V_D \cong R_D / (V_D \cap R_D)$ , by Theorem 1.7 and Lemma 2.1 (iii), we only need to decompose  $\lambda_D|_{V_D \cap R_D}^{R_D}$  instead of decomposing the supercharacter  $\xi_{D, \phi} = \lambda_D^U$ . Hence, all work is restricted to a pattern subgroup of  $U_k(q)$  where  $k = |D| < n$ .

*Proof of Corollary 1.9.* Theorem 1.7 gives a correspondence one-to-one on the multiplicities and degrees between  $\text{Irr}(V_D R_D, \lambda_D)$  and  $\text{Irr}(\xi_{D, \phi})$ , i.e.  $|\text{Irr}(V_D R_D, \lambda_D)| = |\text{Irr}(\xi_{D, \phi})|$ , and if  $\chi \in \text{Irr}(V_D R_D, \lambda_D)$  has multiplicity  $t$  then  $\chi^U \in \text{Irr}(\xi_{D, \phi})$  also has multiplicity  $t$ , and  $\chi^U(1) = [U : V_D R_D]\chi(1)$ . Therefore, it is enough to show that  $\chi \in \text{Irr}(R_D, \lambda_D|_{V_D \cap R_D})$  has multiplicity  $\chi(1)$ .

By Lemma 2.1 (i),  $K_D \cap V_D \subset \ker(\lambda_D|_{V_D \cap R_D}) \cap \ker(\lambda_D|_{V_D \cap R_D}^{R_D})$  is normal in  $R_D$ . So  $\lambda_D|_{V_D \cap R_D}$  can be considered as a linear character of the quotient group  $R_D / (K_D \cap R_D)$ . By Lemma 2.1 (ii),  $(V_D \cap R_D) / (K_D \cap R_D) \subset Z(R_D / (K_D \cap R_D))$ ,  $\lambda_D|_{V_D \cap R_D}$  is a linear character of the center, and the claim holds.  $\square$

### 3. APPLICATIONS

**Sample 1:** The first sample is about supercharacters  $\xi_{D, \phi}$  of  $U_n(q)$  which have only one irreducible constituent, i.e.  $|\text{Irr}(\xi_{D, \phi})| = 1$ . Here, we list two families of them. Without loss of generality, we suppose that  $\alpha_{1, -} \in D$ .

$D_1 = \{\alpha_{1, k}, \alpha_{2, 2k-1}, \alpha_{3, 2k-2}, \dots, \alpha_{k, k+1}, \alpha_{k+1, 2k}\}$  where  $k \geq 2$  and  $2k < n$ . Hence,  $w_{D_1} = \begin{pmatrix} 1 & & \\ & w & \\ & & 1 \end{pmatrix}$  where  $w$  is the longest element in the Weyl group  $S_{k-1}$  of  $GL_{k-1}(q)$ .



$D_2 = \{\alpha_{1,2}, \alpha_{2,3}, \dots, \alpha_{2m-1,2m}\}$  where  $m \geq 1$  and  $2m < n$ , which gives  $w_{D_2}$  equal to the identity  $(2m-1) \times (2m-1)$ -matrix.

By Lemma 1.8,  $R_{D_1}$  is the largest special subgroup  $q^{1+2(k-1)}$  in  $U_{k+1}(q)$ , i.e.  $[R_{D_1}, R_{D_1}] = Z(R_{D_1}) = \Phi(R_{D_1})$  the Frattini subgroup of  $R_{D_1}$ . It is known that a special subgroup of type  $q^{1+2t}$  has  $q^{2t}$  linear characters and  $(q-1)$  almost faithful character of degree  $q^t$ , see Corollary 2.3, [13]. ( $\chi \in \text{Irr}(G)$  is *almost faithful* if  $Z(G) \not\subset \ker(\chi)$ .) Since  $V_{D_1} \cap R_{D_1} = X_{\alpha_{1,k}} = Z(R_{D_1})$ ,  $\lambda_{D_1}|_{X_{\alpha_{1,k}}} \neq 1_{X_{\alpha_{1,k}}}$ . Hence,  $\lambda_{D_1}^{V_{D_1} R_{D_1}}$  has only one irreducible constituent of degree  $q^{k-1}$  with multiplicity  $q^{k-1}$ . By Corollary 1.9,  $\xi_{D_1, \phi}$  has only one constituent of degree  $q^{(k-1)+[U:V_{D_1} R_{D_1}]}$  with multiplicity  $q^{k-1}$ .

We decompose  $\xi_{D_2, \phi}$  by induction on  $m$ . By Corollary 1.9 and Lemma 2.1, we have  $|\text{Irr}(\xi_{D_2, \phi})| = |\text{Irr}(R_{D_2}, \lambda_{D_2}|_{V_{D_2} \cap R_{D_2}})|$ . Let  $D'_2 = \{\alpha_{1,2}, \alpha_{2,3}, \dots, \alpha_{2m-3, 2m-2}\}$ . Since  $w_{D_2} = I_{2m-1}$ , by Lemma 1.8,  $R_{D_2} \simeq U_{2m-1}(q)$ . It is easy to check that  $D'_2 = \{\alpha \in D : X_\alpha \subset R_{D_2}\}$  and  $\lambda_{D_2}|_{V_{D_2} \cap R_{D_2}}^{R_{D_2}} = q \xi_{D'_2, \phi'}$  where  $\xi_{D'_2, \phi'}$  is a supercharacter of  $U_{2m-1}(q)$  with  $\phi' = \phi \setminus \{\lambda_{\alpha_{2m-2, 2m-1}}, \lambda_{\alpha_{2m-1, 2m}}\}$ . Hence, by the hypothesis of induction on  $m$ , it suffices to check  $D_2 = \{\alpha_{1,2}, \alpha_{2,3}, \alpha_{3,4}\}$ , which is  $D_1$  with  $k = 2$ .

**Sample 2:** The second sample is the upper triangular groups  $U_{13}(q)$ . Isaacs-Karagueuzian [12] conjecture that there exists a unique pair of irrational irreducible characters of  $U_{13}(2)$  of degree  $2^{16}$ . This conjecture has been answered by Marberg ([15], Theorem 9.2) by constructing decomposition trees and he finds out the exact supercharacter giving this pair. Evseev ([7], Theorem 1.4) generalizes the result by applying a reduction algorithm process to obtain  $q(q-1)^{13}$  irreducibles of  $U_{13}(q)$  which cannot be constructed by inducing from a linear character of some pattern group. Those characters are called *not well-induced*. Here, we work independently to find out the supercharacter giving these properties.

Let  $D = \{\alpha_{1,4}, \alpha_{2,5}, \alpha_{5,6}, \alpha_{6,7}, \alpha_{7,8}, \alpha_{3,9}, \alpha_{4,10}, \alpha_{8,11}, \alpha_{9,12}\} \subset \Sigma_{12}^+$ . Let  $U = U_{13}(q)$ . By Lemma 1.8, we have

$$w_D = \begin{pmatrix} 1 & & & & & & & & & & & & \\ & 1 & & & & & & & & & & & \\ & & & & & & 1 & & & & & & \\ & & & & & & & 1 & & & & & \\ & & & 1 & & & & & & & & & \\ & & & & 1 & & & & & & & & \\ & & & & & 1 & & & & & & & \\ & & & & & & 1 & & & & & & \\ & & & & & & & 1 & & & & & \\ & & & & & & & & 1 & & & & \\ & & & & & & & & & 1 & & & \end{pmatrix}$$

and

$$R_D \simeq \begin{array}{cccccccc} \alpha_1 & & & & * & \bullet & \bullet & \bullet & \bullet \\ & \alpha_2 & & & & * & \bullet & \bullet & \bullet \\ & & \alpha_3 & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & \alpha_5 & * & \bullet & \bullet \\ & & & & & & \alpha_6 & * & \bullet \\ & & & & & & & \alpha_7 & * \\ & & & & & & & & \alpha_8 \end{array}$$

So  $[U : V_D R_D] = q^{12}$ ,  $\lambda_D^U(1) = q^{27}$ .  $R_D$  is isomorphic to  $U_9(q) \cap {}^{w_D}U_9(q)$  as in the above picture, where  $D \cap \Gamma_D =$  all  $*$ 's,  $K_D \cap R_D =$  all  $X_\bullet$ 's  $\subset \ker(\lambda_D)$ , and  $V_D \cap R_D =$  both  $X_*$ 's and  $X_\bullet$ 's. More exactly, if we let  $D = \{\tau_1, \dots, \tau_9\}$  in ascending order  $<_r$  and set  $\mu = \lambda_D|_{V_D \cap R_D} = \bigotimes_{\tau_i \in D} \lambda_{\tau_i, s_i}|_{V_D \cap R_D}$  where  $s_i \in \mathbb{F}_q^\times$ ,



then  $\mu|_{X_{\alpha_{1,4}}} = \lambda_{\tau_1, s_1}$ ,  $\mu|_{X_{\alpha_{2,5}}} = \lambda_{\tau_2, s_2}$ ,  $\mu|_{X_{\alpha_{5,6}}} = \lambda_{\tau_3, s_3}$ ,  $\mu|_{X_{\alpha_{6,7}}} = \lambda_{\tau_4, s_4}$  and  $\mu|_{X_{\alpha_{7,8}}} = \lambda_{\tau_5, s_5}$ . So we can consider  $\mu$  as a linear character of  $R_D$  where  $R_D \subset U_9(q)$ . Since  $K_D \cap R_D \subset \ker(\mu) \cap \ker(\mu^{R_D})$ , we proceed the induction of  $\mu$  in the quotient group  $R_D/(K_D \cap R_D)$ . For the strategy to find all constituents  $\chi$  of  $\mu^{R_D}$ , we always work with the quotient groups  $R_D/H$  where  $H$  is the largest pattern subgroup contained in  $\ker(\chi)$ . First of all, let  $R$  denote  $R_D/(K_D \cap R_D)$ , and  $V$  as  $(V_D \cap R_D)/(K_D \cap R_D)$ .

**Lemma 3.1.**  $\mu^R$  decomposes into

- (i)  $q^3(4q-3)$  distinct irreducible constituents of degree  $q^3$  and each has multiplicity  $q^3$ ,
- (ii)  $q(q-1)(3q^3+q^2+q-3)$  distinct irreducible constituents of degree  $q^4$  and each has multiplicity  $q^4$ ,
- (iii)  $q^2(q+2)(q-1)^2$  distinct irreducible constituents of degree  $q^5$  and each has multiplicity  $q^5$ .

To deal with extensions for a character of a subgroup  $H$  to a group  $G$ , we use the following property where the proof is quite obvious.

**Lemma 3.2.** Let  $H \leq G$  and  $\lambda \in \text{Irr}(H)$ .

- (i) If  $\lambda$  is linear and  $[G, G] \subset \ker(\lambda)$  then  $\lambda$  is extendible to  $G$ .
- (ii) If  $G \simeq H \rtimes K$  where  $K \trianglelefteq G$  then  $\lambda$  inflates to  $G$ .

*Proof of Lemma 3.1.* Since  $X_{\alpha_{1,3}}X_{\alpha_{3,8}} \subset Z(R)$ ,  $L_1 = VX_{\alpha_{1,3}}X_{\alpha_{3,8}}$  is a pattern subgroup of  $R$ . Hence,  $\mu^{L_1}$  decomposes into  $q^2$  linear characters. Let  $\lambda_1$  be an extension of  $\mu$  to  $L_1$ . We divide into four cases:

- Case 1:  $\lambda_1(X_{\alpha_{1,3}}) \neq \{1\} \neq \lambda_1(X_{\alpha_{3,8}})$ , there are  $(q-1)^2$  such  $\lambda_1$ 's.
- Case 2:  $\lambda_1(X_{\alpha_{1,3}}) \neq \{1\} = \lambda_1(X_{\alpha_{3,8}})$ , there are  $(q-1)$  such  $\lambda_1$ 's.
- Case 3:  $\lambda_1(X_{\alpha_{1,3}}) = \{1\} \neq \lambda_1(X_{\alpha_{3,8}})$ , there are  $(q-1)$  such  $\lambda_1$ 's.
- Case 4:  $\lambda_1(X_{\alpha_{1,3}}) = \{1\} = \lambda_1(X_{\alpha_{3,8}})$ , there is only 1 such  $\lambda_1$ .

Case 1: Let  $L_2 = L_1X_{\alpha_2}X_{\alpha_5}X_{\alpha_7}X_{\alpha_{2,3}}X_{\alpha_{4,7}}X_{\alpha_{4,8}}X_{\alpha_{3,7}}$ . Check directly that  $[L_2, L_2] = \{1\}$ ,  $L_2 \trianglelefteq R$  and  $R = L_2X_{\alpha_1}X_{\alpha_3}X_{\alpha_6}X_{\alpha_8}X_{\alpha_{2,4}}$ . Hence,  $\lambda_1$  extends to  $q^8$  linear characters of  $L_2$ . Let  $\lambda_2$  be an extension of  $\lambda_1$  to  $L_2$ . Check directly that the inertia group  $I_R(\lambda_2) = L_2$ . Hence  $\lambda_2^R \in \text{Irr}(R)$  has degree  $q^5$  and multiplicity  $|\{\lambda_2^x : x \in X_{\alpha_1}X_{\alpha_3}X_{\alpha_6}X_{\alpha_8}X_{\alpha_{2,4}}\}| = q^5$ . And there are  $(q-1)^2q^3$  irreducible constituents of this type.

All technique of extensions, inductions, and counting multiplicities for next cases are the same.

Case 2: Since  $X_{\alpha_{3,8}} \subset \ker(\lambda_1^R)$ , we work with the quotient  $R/X_{\alpha_{3,8}}$ . We have  $X_{\alpha_{4,8}} \subset Z(R)$ . Let  $H_2 = L_1X_{\alpha_{4,8}}$ . Then  $\lambda_1$  extends to  $q$  linears of  $H_2$ . Call  $\eta_2$  an extension of  $\lambda_1$  to  $H_2$ . If  $\eta_2(X_{\alpha_{4,8}}) \neq \{1\}$ , then let  $H_3 = H_2X_{\alpha_2}X_{\alpha_6}X_{\alpha_8}X_{\alpha_{1,2}}X_{\alpha_{2,3}}X_{\alpha_{2,4}}X_{\alpha_{3,7}}$ . Check directly that  $[H_3, H_3] = 1$ ,  $H_3 \trianglelefteq R$  and  $R = H_3X_{\alpha_1}X_{\alpha_3}X_{\alpha_5}X_{\alpha_7}X_{\alpha_{3,7}}$ . Hence,  $\eta_2^{H_3}$  decomposes to  $q^7$  linear characters of  $H_3$ . Let  $\eta_3$  be an extension of  $\eta_2$  to  $H_3$ . Check directly that the inertia group  $I_R(\eta_3) = H_3$ . Hence,  $\eta_3^R \in \text{Irr}(R)$  and there are  $(q-1)^2q^2$  irreducible constituents of degree  $q^5$  with multiplicity  $q^5$ .

Otherwise, if  $\eta_2(X_{\alpha_{4,8}}) = \{1\}$ , then  $X_{\alpha_{4,8}} \subset \ker(\eta_2^R)$ . So we work with  $R/X_{\alpha_{4,8}}$ . Let  $H_3 = H_2X_{\alpha_2}X_{\alpha_6}X_{\alpha_8}X_{\alpha_{1,2}}X_{\alpha_{2,3}}X_{\alpha_{2,4}}X_{\alpha_{3,7}}X_{\alpha_{4,7}}$ . Check directly that  $[H_3, H_3] = 1$ ,  $H_3 \trianglelefteq R$  and  $R = H_3X_{\alpha_1}X_{\alpha_3}X_{\alpha_5}X_{\alpha_7}$ . Hence,  $\eta_2^{H_3}$  decomposes into  $q^8$  linear characters. Let  $\eta_3$  be an extension of  $\eta_2$  to  $H_3$ . Check directly that the inertia group

$I_R(\eta_3) = H_3$ . Hence,  $\eta_3^R \in \text{Irr}(R)$  and there are  $(q-1)q^4$  irreducible constituents of degree  $q^4$  with multiplicity  $q^4$ .

Case 3: Since  $X_{\alpha_{1,3}} \subset \ker(\lambda_1^R)$ , we work with the quotient  $R/X_{\alpha_{1,8}}$ . We have  $X_{\alpha_{1,2}} \subset Z(R)$ . Therefore,  $\lambda_1$  extends to  $N_2 = L_1 X_{\alpha_{1,2}}$ . Call  $\lambda_2$  an extension of  $\lambda_1$  to  $N_2$ . If  $\lambda_2(X_{\alpha_{1,2}}) \neq \{1\}$ , let  $N_3 = N_2 X_{\alpha_1} X_{\alpha_5} X_{\alpha_7} X_{\alpha_{2,3}} X_{\alpha_{3,7}} X_{\alpha_{4,7}} X_{\alpha_{4,8}}$ . Check directly that  $[N_3, N_3] = \{1\}$ ,  $N_3 \trianglelefteq R$  and  $R = N_3 X_{\alpha_2} X_{\alpha_3} X_{\alpha_6} X_{\alpha_8} X_{\alpha_{2,4}}$ . Hence,  $\lambda_2^{N_3}$  decomposes into  $q^7$  linear characters. Let  $\lambda_3$  be an extension of  $\lambda_2$  to  $N_3$ . Check directly that the inertia group  $I_R(\lambda_3) = N_3$ . Hence,  $\lambda_3^R \in \text{Irr}(R)$  and there are  $(q-1)^2 q^2$  irreducible constituents of degree  $q^5$  with multiplicity  $q^5$ .

Otherwise, if  $\lambda_2(X_{\alpha_{1,2}}) = \{1\}$ , then  $X_{\alpha_{1,2}} \subset \ker(\lambda_2)$ . So we work with  $R/X_{\alpha_{1,2}}$ . Let  $N_3 = N_2 X_{\alpha_1} X_{\alpha_2} X_{\alpha_5} X_{\alpha_7} X_{\alpha_{2,3}} X_{\alpha_{3,7}} X_{\alpha_{4,7}} X_{\alpha_{4,8}}$ . Check directly that  $[N_3, N_3] = \{1\}$ ,  $R = N_3 X_{\alpha_3} X_{\alpha_6} X_{\alpha_8} X_{\alpha_{2,4}}$  and  $N_3 \trianglelefteq R$ . Hence,  $\lambda_2^{N_3}$  decomposes into  $q^8$  linear characters. Let  $\lambda_3$  be an extension of  $\lambda_2$  to  $N_3$ . Check directly that the inertia group  $I_R(\lambda_3) = N_3$ . Hence,  $\lambda_3^R \in \text{Irr}(R)$  and there are  $(q-1)q^4$  irreducible constituents of degree  $q^4$  with multiplicity  $q^4$ .

Case 4: Since  $X_{\alpha_{1,3}} X_{\alpha_{3,8}} \subset \ker(\lambda_1^R)$ , we work with  $R/X_{\alpha_{1,3}} X_{\alpha_{3,8}}$ . We have  $X_{\alpha_{1,2}} X_{\alpha_{2,3}} X_{\alpha_{3,7}} X_{\alpha_{4,8}} \subset Z(R)$ . Hence,  $\lambda_1$  extends to  $T_2 = L_1 X_{\alpha_{1,2}} X_{\alpha_{2,3}} X_{\alpha_{3,7}} X_{\alpha_{4,8}}$ . Call  $\mu_2$  an extension of  $\lambda_1$  to  $T_2$ . We consider five subcases whether these four root subgroups are contained in the kernel of  $\mu_2$  or not. It is noted that there are 9 special subgroups of type  $q^{1+2}$  whose centers are contained in  $Z(R)$ ; and for each  $X_\alpha \subset R$  and  $X_\alpha \not\subset Z(R)$ , there are exactly two subgroups  $S_i, S_j$  in those nine specials such that  $S_i \cap S_j = X_\alpha$  as follows:

$$\begin{aligned} Z(R) &= X_{\alpha_{1,2}} X_{\alpha_{2,3}} X_{\alpha_{3,7}} X_{\alpha_{4,8}} X_{\alpha_{7,8}} X_{\alpha_{6,7}} X_{\alpha_{5,6}} X_{\alpha_{2,5}} X_{\alpha_{1,4}} \text{ and} \\ S_1 &= X_{\alpha_1} X_{\alpha_{1,2}} X_{\alpha_2}, & S_2 &= X_{\alpha_2} X_{\alpha_{2,3}} X_{\alpha_3}, & S_3 &= X_{\alpha_3} X_{\alpha_{3,7}} X_{\alpha_{4,7}}, \\ S_4 &= X_{\alpha_{4,7}} X_{\alpha_{4,8}} X_{\alpha_8}, & S_5 &= X_{\alpha_8} X_{\alpha_{7,8}} X_{\alpha_7}, & S_6 &= X_{\alpha_7} X_{\alpha_{6,7}} X_{\alpha_6}, \\ S_7 &= X_{\alpha_6} X_{\alpha_{5,6}} X_{\alpha_5}, & S_8 &= X_{\alpha_5} X_{\alpha_{2,5}} X_{\alpha_{2,4}}, & S_9 &= X_{\alpha_{2,4}} X_{\alpha_{1,4}} X_{\alpha_1}. \end{aligned}$$

Subcase a: All of them are in kernel of  $\mu_2$ . Let  $T_3 = T_2 X_{\alpha_2} X_{\alpha_3} X_{\alpha_{4,7}} X_{\alpha_7} X_{\alpha_5} X_{\alpha_1}$ . Then  $\mu_2$  extends to  $T_3$ . Call  $\mu_3$  an extension of  $\mu_2$  to  $T_3$ . Each  $\mu_3$  induces irreducibly to  $R$  by checking  $I_R(\mu_3) = T_3$ . Therefore, there are  $q^3$  irreducible constituents of degree  $q^3$  and multiplicity  $q^3$ .

Subcase b: Three of them are in  $\ker(\mu_2)$  and the other is not. Without loss of generality, suppose  $X_{\alpha_{1,2}} \not\subset \ker(\mu_2)$ . Let  $T_3 = T_2 X_{\alpha_3} X_{\alpha_7} X_{\alpha_2} X_{\alpha_{2,4}} X_{\alpha_6} X_{\alpha_8}$  and  $[T_3, T_3] = \{1\}$ . Hence,  $\mu_2$  extends to  $T_3$ . Call  $\mu_3$  an extension of  $\mu_2$  to  $T_3$ . And  $\mu_3$  induces irreducibly to  $R$  by checking  $I_R(\mu_3) = T_3$ . Therefore, there are  $4(q-1)q^3$  distinct irreducible constituents of degree  $q^3$  and multiplicity  $q^3$ .

Subcase c: Two of them are in  $\ker(\mu_2)$  and the others are not. There are 6 smaller cases here. Using the same technique, we get  $6(q-1)^2 q$  distinct irreducible constituents of degree  $q^4$  and multiplicity  $q^4$ .

Subcase d: One of them is in  $\ker(\mu_2)$  and the others are not. There are 4 smaller cases here. Without loss of generality, we suppose  $X_{\alpha_{1,2}} \subset \ker(\mu_2)$ . Then  $\mu_2$  extends to  $T_3 = T_2 X_{\alpha_1} X_{\alpha_5} X_{\alpha_7} X_{\alpha_{4,7}} X_{\alpha_2}$ . Call  $\mu_3$  an extension of  $\mu_2$  to  $T_3$ . Each  $\mu_3$  induces irreducibly to  $R$  by checking  $I_R(\mu_3) = T_3$ . Hence, there are  $4(q-1)^3 q$  distinct irreducible constituents of degree  $q^4$  and multiplicity  $q^4$ .

Subcase e: None of them is in  $\ker(\mu_2)$ . Let  $Q_3 = T_2 X_{\alpha_1} X_{\alpha_5} X_{\alpha_7} X_{\alpha_{4,7}}$ ,  $[Q_3, Q_3] = \{1\}$  and  $Q_3 \trianglelefteq R$ . Hence,  $\mu_2$  extends to  $Q_3$ , call  $\mu_3$  an extension of  $\mu_2$  to  $Q_3$ . Denote  $\mu_3|_{X_{\beta_i}} = \phi_{\beta_i, s_i}$  where  $\beta_i \in \{\alpha_{1,2}, \alpha_{2,3}, \alpha_{3,7}, \alpha_{4,8}, \alpha_{7,8}, \alpha_{6,7}, \alpha_{5,6}, \alpha_{2,5}, \alpha_{1,4}\}$  and  $s_i \in \mathbb{F}_q^\times$ . The inertia group  $I_R(\mu_3)$  is generated by  $Q_3$  and  $x(a)$  for all  $a \in \mathbb{F}_q$ ,

where

$$x(a) = x_{\alpha_2}(-\frac{s_{1,4}}{s_{1,2}}a)x_{\alpha_6}(\frac{s_{2,5}}{s_{5,6}}a)x_{\alpha_8}(\frac{s_{2,5}s_{7,8}}{s_{5,6}s_{6,7}}a)x_{\alpha_3}(\frac{s_{2,5}s_{4,8}s_{7,8}}{s_{3,7}s_{5,6}s_{6,7}}a)x_{\alpha_{2,4}}(a).$$

Check that  $[I_R(\mu_3) : Q_3] = q$  and  $[I_R(\mu_3), I_R(\mu_3)] \subset \ker(\mu_3)$  by showing that  $[x_\beta(s), x(a)] \in \ker(\mu_3)$  where  $\beta \in \{\alpha_1, \alpha_5, \alpha_7, \alpha_{4,7}\}$  and  $s, a \in \mathbb{F}_q$ . For example, by using the nilpotent class 2 of  $R$  and applying  $[x, yz] = [x, z][x, y]^z = [x, z][x, y]$ , we have

$$\begin{aligned} [x_{\alpha_1}(s_1), x(a)] &= [x_{\alpha_1}(s_1), x_{\alpha_{2,4}}(a)][x_{\alpha_1}(s_1), x_{\alpha_2}(-\frac{s_{1,4}}{s_{1,2}}a)] \\ &= x_{\alpha_{1,4}}(as_1)x_{\alpha_{1,2}}(-\frac{s_{1,4}}{s_{1,2}}as_1). \end{aligned}$$

Therefore,

$$\begin{aligned} \mu_3([x_{\alpha_1}(s_1), x(a)]) &= \mu_3(x_{\alpha_{1,4}}(as_1)x_{\alpha_{1,2}}(-\frac{s_{1,4}}{s_{1,2}}as_1)) \\ &= \phi_{\alpha_{1,4}, s_{1,4}}(x_{\alpha_{1,4}}(as_1))\phi_{\alpha_{1,2}, s_{1,2}}(x_{\alpha_{1,2}}(-\frac{s_{1,4}}{s_{1,2}}as_1)) \\ &= \phi(s_{1,4}as_1 - s_{1,2}\frac{s_{1,4}}{s_{1,2}}as_1) \\ &= 1. \end{aligned}$$

Hence,  $\mu_3$  extends to  $I_R(\mu_3)$ . Call  $\mu_4$  an extension of  $\mu_3$  to  $I_R(\mu_3)$ . Then  $\mu_4$  induces irreducibly to  $R$ . Therefore, there are  $(q-1)^4q$  distinct irreducible constituents of degree  $q^4$  and multiplicity  $q^4$ .  $\square$

In subcase e, it is noted that  $I_R(\mu_3)$  is not a pattern subgroup, we get  $(q-1)^4q$  irreducible constituents of  $\lambda_D^{V_D R_D}$ . Therefore, by Theorem 1.8,  $\xi_{D,\phi} = \lambda_D^R$  has  $(q-1)^4q$  distinct constituents of degree  $q^{4+12} = q^{16}$ . Since  $|D| = 9$ , there are  $(q-1)^9$  such supercharacters. So totally we have  $(q-1)^{13}q$  not well-induced characters as stated in Evseev [7]. When  $q = 2^f$ ,  $x(a)^2 = x_{\alpha_{2,3}}(\frac{s_{1,4}s_{2,5}s_{4,8}s_{7,8}}{s_{1,2}s_{3,7}s_{5,6}s_{6,7}}a^2) \notin \ker(\mu_4)$  and the order  $o(x(a)) = 4$  for all  $a \in \mathbb{F}_q^\times$ . So there is  $a_0 \in \mathbb{F}_q$  such that  $\mu_4(x(a_0)^2) = -1$ . Hence,  $\mu_4(x(a_0)) = \pm i \in \mathbb{C} \setminus \mathbb{R}$ , i.e.  $\mu_4$  is an irrational linear character of  $I_R(\mu_3)$ . This explains why  $\mu_4^R$  remains irrational, and so does its corresponding irreducible constituent of  $\lambda_D^U$ . This gives a way to construct the pair of irrational irreducible characters of degree  $2^{16}$  of  $U_{13}(2)$ , Isaacs-Karagueuzian's conjecture [12].

#### ACKNOWLEDGEMENT

This paper was presented at Conference on Algebraic Topology, Group Theory and Representation Theory in Skye 2009 dedicated to 60-th birthdays of Professor Ron Solomon and Professor Bob Oliver. The author is grateful to the organizers of the conference.

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